Applied Mathematics Letters 25 (2012) 1094–1097

Contents lists available at SciVerse ScienceDirect

Applied Mathematics Letters

journal homepage: www.elsevier.com/locate/aml

Generalizations and refinements of Hölder's inequality[☆]

Yong-In Kim[a,*], Xiaojing Yang[b]

[a] Department of Mathematics, University of Ulsan, Ulsan, 680-749, Republic of Korea
[b] Department of Mathematics, Tsinghua University, Beijing 100084, China

ARTICLE INFO

Article history:
Received 14 May 2011
Received in revised form 18 March 2012
Accepted 18 March 2012

Keywords:
Hölder's inequality
Refinement
Generalization

ABSTRACT

Some generalizations and refinements of the well-known Hölder's inequality are obtained.

1. Introduction

Let $a_{ij} > 0$ for $1 \leq i \leq n,\ 1 \leq j \leq m$ and $\sum_{j=1}^{m} p_j^{-1} = 1$ with $p_j > 1$ for $j = 1, 2, \ldots, m$. Then it is well-known that the following Hölder's inequality [1]

$$\sum_{i=1}^{n} \prod_{j=1}^{m} a_{ij} \leq \prod_{j=1}^{m} \left(\sum_{i=1}^{n} a_{ij}^{p_j} \right)^{\frac{1}{p_j}} \tag{1}$$

plays an important role in the study of inequalities and in the field of applied mathematics. For example, the well-known Cauchy's inequality

$$\left(\sum_{i=1}^{n} x_i y_i \right)^2 \leq \left(\sum_{i=1}^{n} x_i^2 \right) \left(\sum_{i=1}^{n} y_i^2 \right)$$

is a special case of (1). As for Hölder's inequality, many generalizations and refinements have been obtained so far. See, for example, [1–8] and the references therein. In [8], the first author of the paper has obtained the following refinements and generalizations of the Hölder's inequalities:

Theorem A. *Let $a_{ij} > 0$ for $1 \leq i \leq n,\ 1 \leq j \leq m$ and $\sum_{j=1}^{m} p_j^{-1} = 1$ with $p_j > 1$ for $j = 1, 2, \ldots, m$. Define a function $h : [0, 1] \to (0, +\infty)$ as*

$$h(t) = \prod_{k=1}^{m} \left[\sum_{i=1}^{n} \left(\prod_{j=1}^{m} a_{ij} \right)^{1-t} (a_{ik}^{p_k})^t \right]^{\frac{1}{p_k}}. \tag{2}$$

☆ The second author is supported by NSFC 60973049 and NSFC 61171121.
* Corresponding author.
E-mail addresses: yikim@mail.ulsan.ac.kr, kyiode@hotmail.com (Y.-I. Kim), yangxj@mail.tsinghua.edu.cn (X. Yang).

0893-9659/$ – see front matter © 2012 Elsevier Ltd. All rights reserved.
doi:10.1016/j.aml.2012.03.027

Then for $0 \le t_1 < t_2 < \cdots < t_k \le 1$, we have the following refinement and generalization of the Hölder's inequality (1):

$$h(0) = \sum_{i=1}^{n} \prod_{j=1}^{m} a_{ij} \le h(t_1) \le h(t_2) \le \cdots \le h(t_k) \le h(1) = \prod_{j=1}^{m} \left(\sum_{i=1}^{n} a_{ij}^{p_j} \right)^{\frac{1}{p_j}} . \tag{3}$$

Moreover, the function h satisfies $h(t) > 0$ for all $t \in (0, 1)$ and is convex, that is, $h''(t) \ge 0$ for all $t \in (0, 1)$.

Similarly, for positive functions $f_k \in L^{p_k}((a, b); (0, +\infty))$ with $p_k > 1$ for $k = 1, 2, \ldots, n$ and $\sum_{k=1}^{n} p_k^{-1} = 1$, we define a positive function $g(t)$ for the integral form of Hölder's inequality as

$$g(t) = \prod_{k=1}^{n} \left[\int_a^b \left(\prod_{j=1}^{n} f_j(x) \right)^{1-t} (f_k^{p_k}(x))^t dx \right]^{\frac{1}{p_k}} . \tag{4}$$

Then the integral form of the Hölder's inequality is given as follows:

Theorem B. *Let $g : [0, 1] \to (0, +\infty)$ be defined as in (4). Then for $0 \le t_1 < t_2 < \cdots < t_k \le 1$, we have the following refinement and generalization of the Hölder's inequality for the integral form:*

$$g(0) = \int_a^b \left(\prod_{k=1}^{n} f_k(x) \right) dx \le g(t_1) \le \cdots \le g(t_k) \le g(1) = \prod_{k=1}^{n} \left(\int_a^b f_k^{p_k}(x) dx \right)^{\frac{1}{p_k}} . \tag{5}$$

Moreover, the function g satisfies $g(t) > 0$ for all $t \in (0, 1)$ and is convex, that is, $g''(t) \ge 0$ for all $t \in (0, 1)$.

Now, the main results of this paper are given in the following:

Theorem 1. *Let $a_{ij} > 0$ for $1 \le i \le n$, $1 \le j \le m$ and $p_j > 1$ for $j = 1, 2, \ldots, m$ and $\sum_{j=1}^{m} p_j^{-1} = 1$. Define a function $H : [a, b] \to (0, +\infty)$, where $[a, b]$ may be $(-\infty, b]$, $[a, +\infty)$ or $(-\infty, +\infty)$, by*

$$H(t) = \prod_{k=1}^{m} \left[\sum_{i=1}^{n} \prod_{j=1}^{m} a_{ij}^{\alpha_{kj}(t)} \right]^{\frac{1}{p_k}} , \tag{6}$$

where $\alpha_{kj} \in C^1([a,\ b])$ for $k, j = 1, 2, \ldots, m$. Assume that H satisfies

$$H(a) = \sum_{i=1}^{n} \prod_{j=1}^{m} a_{ij} \quad \text{and} \quad H(b) = \prod_{j=1}^{m} \left(\sum_{i=1}^{n} a_{ij}^{p_j} \right)^{\frac{1}{p_j}} , \tag{7}$$

and that $H'(t) \ge 0$ for $t \in (a, b)$, where

$$H'(t) = H(t) \cdot \sum_{k=1}^{m} \sum_{i=1}^{n} \sum_{j=1}^{m} \frac{\left(\prod_{j=1}^{m} a_{ij}^{\alpha_{kj}(t)} \right) \ln a_{ij}}{p_k \left(\sum_{i=1}^{n} \prod_{j=1}^{m} a_{ij}^{\alpha_{kj}(t)} \right)} \alpha'_{kj}(t). \tag{8}$$

Then for $a \le t_1 < t_2 < \cdots < t_k \le b$, we have the following refinement and generalization of the Hölder's inequality (1):

$$H(a) = \sum_{i=1}^{n} \prod_{j=1}^{m} a_{ij} \le H(t_1) \le H(t_2) \le \cdots \le H(t_k) \le H(b) = \prod_{j=1}^{m} \left(\sum_{i=1}^{n} a_{ij}^{p_j} \right)^{\frac{1}{p_j}} . \tag{9}$$

Theorem 2. *Assume that the functions $f_k \in L^{p_k}((A, B); (0, +\infty))$ with $-\infty < A < B < +\infty$ for $k = 1, 2, \ldots, n$ and that $p_k > 1$ for $k = 1, 2, \ldots, n$ and $\sum_{k=1}^{n} p_k^{-1} = 1$. Define a function $G : [a,\ b] \to (0, +\infty)$, where $[a, b]$ may be $(-\infty, b]$, $[a, +\infty)$ or $(-\infty, +\infty)$, by*

$$G(t) = \prod_{k=1}^{n} \left[\int_A^B \left(\prod_{j=1}^{n} f_j^{\beta_{kj}(t)}(x) \right) dx \right]^{\frac{1}{p_k}} , \tag{10}$$

where $\beta_{kj} \in C^1([a,\ b])$ for $k, j = 1, 2, \ldots, n$. Assume that G satisfies

$$G(a) = \int_A^B \left(\prod_{j=1}^n f_j(x)\right) dx, \qquad G(b) = \prod_{j=1}^n \left(\int_A^B f_j^{p_j}(x)dx\right)^{\frac{1}{p_j}}, \tag{11}$$

and that $G'(t) \geq 0$ for $t \in (a, b)$, where

$$G'(t) = G(t) \cdot \sum_{k=1}^n \sum_{j=1}^n \frac{\int_A^B \left(\prod_{j=1}^n f_j^{\beta_{kj}(t)}(x)\right) \ln f_j(x)dx}{p_k \int_A^B \left(\prod_{j=1}^n f_j^{\beta_{kj}(t)}(x)\right) dx} \beta'_{kj}(t). \tag{12}$$

Then for $a \leq t_1 < t_2 < \cdots < t_k \leq b$, we have

$$G(a) = \int_A^B \left(\prod_{j=1}^n f(x)\right) dx \leq G(t_1) \leq \cdots \leq G(t_k) \leq G(b) = \prod_{j=1}^n \left(\int_A^B f_j^{p_j}(x)dx\right)^{\frac{1}{p_j}}. \tag{13}$$

2. Proofs of main results

Proof of Theorem 1. It is clear that $H(t) > 0$ for $t \in (a, b)$. Let $b_{ik}(t) = \prod_{j=1}^m a_{ij}^{\alpha_{kj}(t)}$ for $1 \leq k \leq m,\ 1 \leq i \leq n$ and $t \in [a, b]$. Then we have

$$\ln H(t) = \sum_{k=1}^m \frac{1}{p_k} \ln \left[\sum_{i=1}^n b_{ik}(t)\right].$$

Hence for $t \in (a,\ b)$, we have

$$\frac{H'(t)}{H(t)} = \sum_{k=1}^m \sum_{i=1}^n \sum_{j=1}^m \frac{b_{ik}(t) \ln a_{ij}}{p_k \sum_{i=1}^n b_{ik}(t)} \alpha'_{kj}(t),$$

which is equivalent to (8). We divide the rest of the proof of Theorem 1 into the following subcases:

Case 1. Let $[a,\ b] = [0, 1]$. We define $\alpha_{kj}(t) = 1 - t$ if $k \neq j$ and $\alpha_{kj}(t) = 1 - t + p_k t$ if $k = j$. Then it is easy to see that $H(t) = h(t)$ given by (2) in Theorem A. Hence, the result (3) in Theorem A is a special case of Theorem 1, that is, Case 1.

Case 2. Assume that $-\infty < a < b < +\infty$. Let $\tau = \frac{t-a}{b-a}$. Then $\tau(a) = 0, \tau(b) = 1$ with $\frac{d\tau}{dt} = \frac{1}{b-a} > 0$. We define $\overline{\alpha}_{kj}(t) = \alpha_{kj}(\tau) = 1 - \tau = \frac{b-t}{b-a}$ if $k \neq j$, and $\overline{\alpha}_{kj}(t) = \alpha_{kj}(\tau) = 1 - \tau + p_k\tau = \frac{b-t+p_k(t-a)}{b-a}$ if $k = j$. Then $\overline{\alpha}'_{kj}(t) = \frac{1}{b-a}\alpha'_{kj}(\tau)$. In this case, if we replace $\alpha_{kj}(t)$ in Theorem 1 by $\overline{\alpha}_{kj}(t)$, then we see that $H(t) = h(\tau) = h\left(\frac{t-a}{b-a}\right)$ and for $t \in (a,\ b)$,

$$H'(t) = h'(\tau)\frac{d\tau}{dt} = \left(\frac{1}{b-a}\right) h'\left(\frac{t-a}{b-a}\right) \geq 0.$$

Case 3. Assume that $a = -\infty < b < +\infty$. Let $\tau = e^{t-b}$. Then $\tau(a) = \tau(-\infty) = 0, \tau(b) = 1$ with $\frac{d\tau}{dt} = e^{t-b} > 0$. We define $\overline{\alpha}_{kj}(t) = \alpha_{kj}(\tau) = 1 - \tau = 1 - e^{t-b}$ if $k \neq j$, and $\overline{\alpha}_{kj}(t) = \alpha_{kj}(\tau) = 1 - \tau + p_k\tau = 1 + (p_k - 1)e^{t-b}$ if $k = j$. Then $\overline{\alpha}'_{kj}(t) = e^{t-b}\alpha'_{kj}(\tau)$. In this case, if we replace $\alpha_{kj}(t)$ in Theorem 1 by $\overline{\alpha}_{kj}(t)$, then we see that $H(t) = h(\tau) = h(e^{t-b})$ and for $t \in (a, b) = (-\infty, b)$,

$$H'(t) = h'(\tau)\frac{d\tau}{dt} = e^{t-b}h'(e^{t-b}) \geq 0.$$

Case 4. Assume that $-\infty < a < b = +\infty$. Let $\tau = 1 - e^{a-t}$. Then $\tau(a) = 0, \tau(b) = \tau(+\infty) = 1$ with $\frac{d\tau}{dt} = e^{a-t} > 0$. We define $\overline{\alpha}_{kj}(t) = \alpha_{kj}(\tau) = 1 - \tau = e^{a-t}$ if $k \neq j$, and $\overline{\alpha}_{kj}(t) = \alpha_{kj}(\tau) = 1 - \tau + p_k\tau = (1 - p_k)e^{a-t} + p_k$ if $k = j$. Then $\overline{\alpha}'_{kj}(t) = e^{a-t}\alpha'_{kj}(\tau)$. In this case, if we replace $\alpha_{kj}(t)$ in Theorem 1 by $\overline{\alpha}_{kj}(t)$, then we see that $H(t) = h(\tau) = h(1 - e^{a-t})$ and for $t \in (a, b) = (a, +\infty)$,

$$H'(t) = h'(\tau)\frac{d\tau}{dt} = e^{a-t}h'(1 - e^{a-t}) \geq 0.$$

Case 5. Assume that $a = -\infty$ and $b = +\infty$. Let $\tau = \frac{e^t}{1+e^t}$. Then $\tau(a) = \tau(-\infty) = 0, \tau(b) = \tau(+\infty) = 1$ with $\frac{d\tau}{dt} = \frac{e^t}{(1+e^t)^2} > 0$. We define $\overline{\alpha}_{kj}(t) = \alpha_{kj}(\tau) = 1 - \tau = \frac{1}{1+e^t}$ if $k \neq j$, and $\overline{\alpha}_{kj}(t) = \alpha_{kj}(\tau) = 1 - \tau + p_k\tau = \frac{1+p_k e^t}{1+e^t}$ if $k = j$.

Then $\overline{\alpha}'_{kj}(t) = \frac{e^t}{(1+e^t)^2}\alpha'_{kj}(\tau)$. In this case, if we replace $\alpha_{kj}(t)$ in Theorem 1 by $\overline{\alpha}_{kj}(t)$, then we see that $H(t) = h(\tau) = h\left(\frac{e^t}{1+e^t}\right)$ and for $t \in (a, b) = (-\infty, +\infty)$,

$$H'(t) = h'(\tau)\frac{d\tau}{dt} = \frac{e^t}{(1+e^t)^2}h'\left(\frac{e^t}{1+e^t}\right) \geq 0.$$

The rest of the proof is obvious. This completes the proof of Theorem 1. □

Proof of Theorem 2. Similarly, we see that $G(t) > 0$ for $t \in (a, b)$. Let

$$F_k(t) = \int_A^B \left(\prod_{j=1}^n f_j^{\beta_{kj}(t)}(x)\right) dx, \quad k = 1, 2, \ldots, n.$$

Then we have

$$\ln G(t) = \sum_{k=1}^n \frac{1}{p_k} \ln F_k(t).$$

Hence for $t \in (a, b)$, we have

$$\begin{aligned} \frac{G'(t)}{G(t)} &= \sum_{k=1}^n \frac{1}{p_k}\frac{F'_k(t)}{F_k(t)} \\ &= \sum_{k=1}^n \sum_{j=1}^n \frac{1}{p_k} \frac{\int_A^B \left(\prod_{j=1}^n f_j^{\beta_{kj}(t)}(x)\right) \ln f_j(x) dx}{F_k(t)} \beta'_{kj}(t), \end{aligned}$$

which is equivalent to (12). □

References

[1] E.F. Beckenbach, R. Bellman, Inequalities, Springer Verlag, Berlin, 1961.
[2] P. Bégout, F. Soria, A generalized interpolation inequality and its application to the stabilization of damped equations, J. Differential Equations 240 (2) (2007) 324–356.
[3] J.A. Carroll, R. Cordner, C.J.A. Evelyn, A new extension of Hölder's inequality, Enseign. Math. 16 (1970) 69–71.
[4] D.E. Daykin, C.J. Eliezer, Generalization of Hölder's and Minkowski's inequalities, Proc. Cambridge Philos. Soc. 64 (1968) 1023–1027.
[5] G.H. Harddy, J.E. Littlewood, G. Polya, Inequalities, Cambridge University Press, 1952.
[6] W.S. He, Generalization of a sharp Hölder's inequality and its application, J. Math. Anal. Appl. 332 (2007) 741–750.
[7] D.S. Mitrinovic, J.E. Pecaric, On an extension of Hölder's inequality, Boll. Unione Mat. Ital. 4-A (7) (1990) 405–408.
[8] X. Yang, Hölder's inequality, Appl. Math. Lett. 16 (2003) 897–903.